Bayes' Theorem From Partial Moments

Viole, Fred
fred.viole@gmail.com

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This quick note is intended to further inform the reader on the inherent flexibility of partial moments. Such flexibility is also exemplified in the following examples, and the reader is strongly urged to recreate them in order to grasp a feel for the wide-ranging potential of the NNS R package.
https://github.com/OVVO-Financial/NNS/tree/NNS-Beta-Version/examples
Introductory material on partial moments and their role as the elements of variance is available in the following posts:
https://www.linkedin.com/today/author/0_1vmA3FLtiRRbU27shcPrMU

Bayes' Theorem

Bayes' Theorem is represented by:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

and we will show the corresponding partial moment equivalences to each of the probabilities.

The following example is used after installing NNS:

```
> require(devtools)
> install_github("OVVO-Financial/NNS", ref = "NNS-Beta-Version")
> require(NNS)
> data <- read.csv("https://goo.gl/Hd2NYR", header = T, sep = "\t")
```

Data

The data consists of 100 observations of S&P 500 and 10-Year yield returns during the period of 2012-05-08 :: 2012-09-27 and orginally appeared in Viole and Nawrocki (2013), "Causation" available at: https://ssrn.com/abstract=2273756
S&P 500 data (S):

```
> S=data$sp;S
```

```
  [1]  0.0256 -0.0150  0.0153 -0.0040 -0.0258  0.0118  0.0201  0.0165  0.0017
 [10]  0.0013 -0.0281  0.0374  0.0198  0.0131 -0.0016  0.0133  0.0065 -0.0094
 [19] -0.0290  0.0057  0.0211  0.0235  0.0340  0.0184  0.0198  0.0054  0.0144
 [28] -0.0265  0.0395  0.0319  0.0022  0.0041 -0.0444  0.0288  0.0280 -0.0508
 [37]  0.0108 -0.0703 -0.0175 -0.0284  0.0398  0.0236 -0.0452 -0.0646  0.0190
 [46] -0.0516 -0.2281 -0.0927 -0.0062 -0.0137 -0.0723 -0.0616  0.1135  0.0620
 [55]  0.0259  0.0104  0.0760  0.0339  0.0219  0.0189  0.0203  0.0118 -0.0311
 [64]  0.0561  0.0385 -0.0622 -0.0378 -0.0033  0.0069  0.0315  0.0432  0.0230
 [73]  0.0349  0.0326  0.0296 -0.0127  0.0205  0.0051 -0.0389  0.0290 -0.1115
 [82] -0.0097  0.0280  0.0158  0.0137  0.0450  0.0391  0.0268 -0.0020 -0.0331
 [91] -0.0134  0.0271  0.0316  0.0281 -0.0039 -0.0306  0.0197  0.0400  0.0213
[100]  0.0252
```

10-Year yield data (I):

```
> I=data$ten.yr;I
```

```
  [1]  0.0095 -0.0024 -0.0119  0.0762 -0.0362 -0.0472 -0.0344  0.0440  0.0190
 [10] -0.0142  0.0601  0.0178 -0.0155 -0.0112  0.0334  0.0323  0.0556  0.0238
 [19]  0.0000 -0.0039 -0.0421 -0.0333  0.0021 -0.0279 -0.0087  0.0429 -0.0084
 [28] -0.0345  0.0281  0.0127  0.0711 -0.0198 -0.0683 -0.0326  0.0022 -0.0876
 [37] -0.0121 -0.0919  0.0000 -0.0635  0.0473  0.0529  0.0552 -0.0222 -0.0304
 [46] -0.0528  0.0320 -0.0763 -0.3775  0.0405  0.1301 -0.0176  0.0383  0.1159
 [55]  0.1228 -0.0440  0.0084 -0.0544 -0.0029  0.0029  0.0544  0.0383 -0.0108
 [64]  0.0108  0.0317 -0.1184 -0.0665 -0.0612 -0.1087 -0.0187 -0.0424  0.0831
 [73]  0.1757  0.0299  0.0545 -0.0487  0.0146 -0.0875 -0.0551  0.0000 -0.2657
 [82] -0.1498  0.0824 -0.0673 -0.0150 -0.0051  0.0000  0.0967 -0.0569 -0.1301
 [91] -0.1054 -0.0572  0.0935  0.0235  0.0173 -0.0588  0.0415  0.1048  0.0360
[100] -0.0102
```

Question for Bayes

The question we seek to answer in this elementary example is "What is the probability of a stock market increase given an increase in interest rates?". Bayes' theorem is perfectly suited to answer such a query.

Notations

Bayes:

P(SI) = probability of the SP 500 increasing
P(SD) = probability of the SP 500 decreasing
P(II) = probability of interest rates increasing
P(ID) = probability of interest rates decreasing

$$P(SI|II) = \frac{P(II|SI)P(SI)}{P(II)}$$

Partial Moments:

$SP500 \leq target, InterestRates \leq target \rightarrow CLPM$ matrix
$SP500 \leq target, InterestRates > target \rightarrow DUPM$ matrix
$SP500 > target, InterestRates \leq target \rightarrow DLPM$ matrix
$SP500 > target, InterestRates > target \rightarrow CUPM$ matrix

Since we are examining increases and decreases, $target = 0$ in this example, but it can accomodate any value of interest.

Data Tables

Simply counting the events in the data above yields the following outcomes:

	Interest increase	Interest Decrease	Unch	Total
Stock Increase	35	28	2	65
Stock Decrease	9	24	2	35
Unch				
Total	44	52	4	100

Partial moments represent the following entries (R commands in red), where the rows represent the A variable and the columns represent the given B variable in the general form:

$$PM(degree.x, degree.y, X, Y, target.x, target.y)$$

S↓ $I \rightarrow$	Interest Increase	Interest Decrease	Unch	Total
Stock Increase	Co.UPM(0,0,S,I,0,0)	D.LPM(0,0,S,I,0,0)		UPM(0,0,S)
Stock Decrease	D.UPM(0,0,S,I,0,0)	Co.LPM(0,0,S,I,0,0)		LPM(0,0,S)
Unch				
Total	UPM(0,0,I)	LPM(0,0,I)		UPM+LPM=1

```
> SI.II = matrix(c(Co.UPM(0, 0, S, I, 0, 0), D.LPM(0, 0, S, I,
    0, 0), UPM(0, 0, S), D.UPM(0, 0, S, I, 0, 0), Co.LPM(0, 0,
    S, I, 0, 0), LPM(0, 0, S), UPM(0, 0, I), LPM(0, 0, I), 1),
    byrow = TRUE, nrow = 3)
> colnames(SI.II) = c("Interest Increase", "Interest Decrease",
    "Total")
> rownames(SI.II) = c("Stock Increase", "Stock Decrease", "Total")
> SI.II
```

```
               Interest Increase Interest Decrease Total
Stock Increase              0.35              0.28  0.65
Stock Decrease              0.09              0.24  0.35
Total                       0.44              0.56  1.00
```

Note how the unchanged entries do not affect the co-partial moments.

Probailities

Probability of a Stock Increase

$P(SI) = \frac{65}{100}$

which is the same as the degree 0 UPM of the S&P 500 from a 0 target:

```
> UPM(0,0,S)

[1] 0.65
```

Probability of a Interest Rate Increase

$P(II) = \frac{44}{100}$

which is the same as the degree 0 UPM of the 10-Year yield from a 0 target:

```
> UPM(0,0,I)

[1] 0.44
```

Probability of a Interest Rate Increase Given a Stock Increase

$P(II|SI) = \frac{35}{65} = 0.5385$

which is identical to the degree 0 co-UPM of the S&P 500 and the 10-Year yield from a 0 target, divided by the probability of an increase in the S&P 500, defined above as $UPM(0,0,S)$.

```
> Co.UPM(0,0,S,I,0,0)/UPM(0,0,S)

[1] 0.5384615
```

Putting it all together

Combining all of our probabilities, we can now answer our question using Bayes' theorem.

$$P(SI|II) = \frac{\frac{35}{65} * \frac{65}{100}}{\frac{44}{100}} = \frac{35}{44} = 0.7954545$$

and via partial moments:

$$P(SI|II) = \frac{\frac{Co.UPM(0,0,S,I,0,0)}{UPM(0,0,S)} * UPM(0,0,S)}{UPM(0,0,I)}$$

```
> ((Co.UPM(0,0,S,I,0,0)/UPM(0,0,S))*UPM(0,0,S))/
   UPM(0,0,I)
```

```
[1] 0.7954545
```

which simplifies greatly (as does Bayes) to just $\frac{Co.UPM(0,0,S,I,0,0)}{UPM(0,0,I)}$

```
> Co.UPM(0,0,S,I,0,0)/UPM(0,0,I)
```

```
[1] 0.7954545
```

Additional Information

Furthermore, partial moments are aptly suited to provide an answer to a more motivated question, "How much on average do stocks increase given an interest rate increase?" All we have to do is alter the degree of the dependent variable (S), and subtract the negative instances.

This is precisely the inverse of an expected shortfall calculation, which is the average loss beyond a VaR % of interest.

We subtract the degree (1|0) D.UPM(S,I) from the degree (1|0) Co.UPM(S,I) and divide that by the probability of an interest rate increase already defined as $UPM(0, 0, I)$.

```
> (Co.UPM(1, 0, S, I, 0, 0) - D.UPM(1, 0, S, I, 0, 0))/UPM(0, 0,
     I)
```

```
[1] 0.01495909
```

Alternatively, we can use some convenient subsetting features of R to reduce the calculation to univariate partial moments:

```
> UPM(1,0,S[I>0])-LPM(1,0,S[I>0])
```

```
[1] 0.01495909
```

and since we know $UPM(1, 0, x) - LPM(1, 0, x) = \bar{x}$
(see here for equivalence) this can be reduced to:

```
> mean(S[I>0])
```

```
[1] 0.01495909
```

Canonical Breast Cancer Example

We can also recreate the widely used breast cancer example when people explain Bayes' theorem.

- 1% of women have breast cancer (and therefore 99% do not).

- 80% of mammograms detect breast cancer when it is there (and therefore 20% miss it).

- 10% of mammograms detect breast cancer when it's not there (and therefore 90% correctly return a negative result).

We can create 2 variables representing 1000 individuals:

- C where a 1 will denote those with cancer and -1 for those without cancer.
- Test where a 1 will denote those with a positive test result and a -1 for those with a negative test result.

```
> C=c(rep(1,8),rep(-1,990),rep(1,2)); Test=c(rep(1,107),rep(-1,893))
```

Data Tables

We can assemble the probabilities in the data above in the following table:

	Cancer	No Cancer	Total
Test Positive	(80% * 1%) = .008	(10% * 99%) = .099	.107
Test Negative	(20% * 1%) = .002	(90% * 99%) = .891	.893
Total	.01	.99	1

Partial moments represent the following entries (R commands in red), where this time the columns represent the A variable and the rows represent the given B variable:

Test$\downarrow$ $C \rightarrow$	Cancer	No Cancer	Total
Test Positive	Co.UPM(0,0,Test,C,0,0)	D.LPM(0,0,Test,C,0,0)	UPM(0,0,Test)
Test Negative	D.UPM(0,0,Test,C,0,0)	Co.LPM(0,0,Test,C,0,0)	LPM(0,0,Test)
Total	UPM(0,0,C)	LPM(0,0,C)	UPM+LPM=1

```
> C.Test = matrix(c(Co.UPM(0, 0, Test, C, 0, 0), D.LPM(0, 0, Test,
    C, 0, 0), UPM(0, 0, Test), D.UPM(0, 0, Test, C, 0, 0), Co.LPM(0,
    0, Test, C, 0, 0), LPM(0, 0, Test), UPM(0, 0, C), LPM(0,
```

```
  0, C), 1), byrow = TRUE, nrow = 3)
> colnames(C.Test) = c("Cancer", "No Cancer", "Total")
> rownames(C.Test) = c("Test Positive", "Test Negative", "Total")
> C.Test
```

```
              Cancer No Cancer Total
Test Positive  0.008     0.099 0.107
Test Negative  0.002     0.891 0.893
Total          0.010     0.990 1.000
```

So what's the chance we really have cancer given a positive test result?

$$P(C|Test) = \frac{P(Test|C) * P(C)}{P(Test)}$$

Putting it all together

Combining all of our probabilities, we can now answer our question using Bayes' theorem.

$$P(C|Test) = \frac{\frac{.008}{.01} * .01}{.107} = \frac{0.008}{.107} = 0.074766355$$

and via partial moments:

$$P(C|Test) = \frac{\frac{Co.UPM(0,0,C,Test,0,0)}{UPM(0,0,C)} * UPM(0,0,C)}{UPM(0,0,Test)}$$

```
> ((Co.UPM(0,0,C,Test,0,0)/UPM(0,0,C))*UPM(0,0,C))/
   UPM(0,0,Test)
```

```
[1] 0.07476636
```

which simplifies greatly (as does Bayes) to just $\frac{Co.UPM(0,0,C,Test,0,0)}{UPM(0,0,Test)}$

```
> Co.UPM(0,0,C,Test,0,0)/UPM(0,0,Test)
```

```
[1] 0.07476636
```

Summary

We are not re-inventing a wheel. Conditional probability is conditional probability, nothing has changed. What we have (uniquely ?) done is represent the partial moment equivalences to these conditional probabilities. Given the myriad of other classical equivalences partial moments demonstrate, the only surprise is the lack of consideration they receive.